Effect of m_c on b quark chromomagnetic interaction and on-shell two-loop integrals with two masses

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The effect of a nonzero c quark mass on the b quark HQET Lagrangian, up to the $1/m_b$ level, is calculated at two loops. The results are expressed in terms of dilogarithmic functions of m_c/m_b. This calculation involves on-shell two-loop propagator-type diagrams with two different masses m_b and m_c. A general algorithm for reducing such Feynman integrals to the basis of two nontrivial and two trivial integrals is constructed. [S0556-2821(99)05403-X]

I. INTRODUCTION

On-shell two-loop calculations in any theory containing two massive fields with different masses and a massless field necessarily involve diagrams such as Fig. 1. Such diagrams appear, for example, in QED with e, μ and τ, in QCD with b and c quarks, and in the electroweak theory. In general, such diagrams have a two-particle threshold[1] at $p^2 = M^2$, and a three-particle threshold at $p^2 = (M+2m)^2$, where p is the external momentum. There are two three-particle pseudothresholds, at $p^2 = (M-2m)^2$ and $p^2 = M^2$. Therefore, the on-shell condition $p^2 = M^2$ means that we are, at the same time, at the two-particle (pseudo)threshold and at the three-particle pseudothreshold.

Combining the identical massless denominators, we can express these diagrams via scalar integrals (see Fig. 2)

$$I(n_1,n_2,n_3,n_4) \equiv -\frac{1}{\pi^d} \int \frac{d^d k \, d^d l}{\mathcal{D}_1^{n_1} \mathcal{D}_2^{n_2} \mathcal{D}_3^{n_3} \mathcal{D}_4^{n_4}} \Bigg|_{p^2 = M^2}, \tag{1.1}$$

where

$$\mathcal{D}_1 = M^2 - (p+k)^2, \quad \mathcal{D}_2 = -k^2,$$

$$\mathcal{D}_3 = m^2 - l^2, \quad \mathcal{D}_4 = m^2 - (k-l)^2, \tag{1.2}$$

and $d = 4 - 2\varepsilon$ is the space-time dimension (in the framework of dimensional regularization [1]). Evidently,

$$I(n_1,n_2,n_4,n_3) = I(n_1,n_2,n_3,n_4).$$

Some integrals of this class have been already considered in the literature [2–8].[2] Here we propose a general algorithm, based on integration by parts [11], which reduces any integral (1.1) (with integer n_i) to the basis of two nontrivial integrals (known near four dimensions up to finite terms in ε) and two trivial ones.

In realistic calculations, we also get integrals (1.1) with scalar numerators. In some cases, these numerators cannot be directly expressed in terms of the denominators (1.2). Such integrals are denoted as $\tilde{I}(n_1,n_2,n_4,n_3;n_0)$ [where n_0 is the power of the corresponding numerator, see Eqs. (2.15),(2.16)], and they are considered in Sec. II. They can be reduced to the same basis as the integrals $I(n_1,n_2,n_4,n_3)$.

The integration-by-parts technique [11] was first developed in the context of massless calculations. Later it was used in on-shell two-loop calculations with a single mass [3,4,12,13] as well as in heavy quark effective theory (HQET) [14] (see a short review [15]).

Recently it was shown [16] (cf. also [17]) that any two-loop propagator integral with generic masses and external momentum can be reduced to a finite set of basis integrals.[3] However, in special cases such as thresholds and pseudothresholds some general formulas of Ref. [16] become degenerate. Therefore, these cases require special examination. In particular, here we propose a reduction algorithm for the integrals (1.1) and implement it as a REDUCE [19,20] package.

Integrals of this class appear in matching QCD to HQET (see [21] for review and references). Specifically, let us consider the b quark HQET with $m_b \equiv M$, keeping one more massive flavor, c, with $m_c \equiv m$. The coefficients in the HQET Lagrangian (as well as in the $1/M$ expansion of QCD operators) are derived by equating the on-shell matrix elements in QCD and HQET. In order to determine the coefficients in the Lagrangian up to the $1/M$ level, it is sufficient to equate scattering amplitudes of an on-shell heavy quark in an external gluon field:

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[1]It coincides with the corresponding pseudothreshold, since one of the particles involved is massless.

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[2]The asymptotic expansion of similar on-shell integrals has been constructed in [9,10]

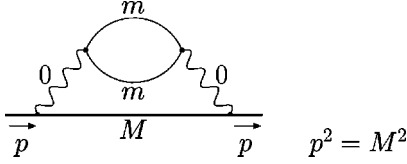[3]The algorithm of [16] was originally implemented in FORM and then [18] in MATHEMATICA.

FIG. 1. Two-loop self-energy diagram.

$$\bar{u}(p')t^a\left(F_1(q^2)\frac{(p+p')^\mu}{2M}+G_m(q^2)\frac{[\slashed{q},\gamma^\mu]}{4M}\right)u(p)$$

$$=\bar{u}_v(q)t^a\left(v^\mu+\frac{q^\mu}{2M}+C_m(\mu)\tilde{Z}^{-1}(\mu)\tilde{\mu}_{g0}\frac{[\slashed{q},\gamma^\mu]}{4M}\right)$$

$$\times u_v(0)+\mathcal{O}\left(\frac{q^2}{M^2}\right). \tag{1.3}$$

Here $p=Mv$, $q=p'-p$, $p^2=p'^2=M^2$; form factors on the QCD side are $F_1(q^2)=1+\mathcal{O}(q^2)$, $G_m(q^2)=\mu_g+\mathcal{O}(q^2)$; $C_m(\mu)$ is the coefficient of the chromomagnetic interaction, the only nontrivial coefficient in the HQET Lagrangian to this order; $\tilde{Z}^{-1}(\mu)\tilde{\mu}_{g0}$ is the matrix element of the renormalized HQET chromomagnetic operator [note that $C_m(\mu)\tilde{Z}^{-1}(\mu)$ does not depend on the normalization scale μ]; the spinors are related by the Foldy-Wouthuysen transformation $u(Mv+k)=[1+\slashed{k}/(2M)+\mathcal{O}(k^2/M^2)]u_v(k)$ (see [22] for a recent short review of the status of calculations of coefficients in the HQET Lagrangian). Two-loop anomalous dimension of the HQET chromomagnetic operator (or $\tilde{Z}$) was calculated in [23,24]. The chromomagnetic coefficient $C_m(\mu\sim M)$ was calculated at two loops in [24], under the assumption that all other flavors (except the heavy flavor of HQET) are massless; higher orders in the large β_0 limit were summed in [25].

Here we calculate the effect of nonzero c quark mass on the chromomagnetic interaction in the b quark HQET. The HQET on-shell loop integrals without massive quark loops contain no scale and hence vanish. Therefore, only the integrals involving c quark loops are relevant. A simple method for calculating such integrals was proposed in [5] (and used in [24]). The QCD calculation involves the integrals (1.1). The nonzero m_c effect on heavy-light bilinear quark currents in HQET was considered in [5].

II. ON-SHELL TWO-LOOP INTEGRALS WITH TWO MASSES

Integrals (1.1) with $n_3\leq0$ or $n_4\leq0$ reduce to products of one-loop ones, and are proportional to $T_1\equiv I(1,0,1,0)$ with rational coefficients. Integrals with $n_1\leq0$ are two-loop vacuum ''bubbles,'' an explicit formula for them can be found in [3]; they are proportional to $T_0\equiv I(0,0,1,1)$ with rational coefficients. These trivial basis integrals are

$$T_0=(m^2)^{d-2}\Gamma^2(1-d/2), \quad T_1=(Mm)^{d-2}\Gamma^2(1-d/2). \tag{2.1}$$
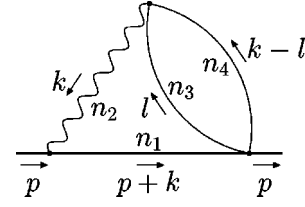

FIG. 2. On-shell two-loop integral with two masses.

Explicit general formulas for similar integrals with numerators can be found in Ref. [26].[4]

When n_1, n_3, and n_4 are all positive, we employ the integration-by-parts technique [11]. Applying the operator $(\partial/\partial k)\cdot(k-l)$ to the integrand of Eq. (1.1) and substituting $p\cdot l\to(p\cdot k)(k\cdot l)/k^2$ in the numerator, we obtain the recurrence relation

$$4m^2n_4\mathbf{4}^+I=[-2d+n_1+2n_2+4n_4-(n_1+2n_2)(\mathbf{3}^--\mathbf{4}^-)$$
$$\times\mathbf{2}^++n_1(\mathbf{2}^--\mathbf{3}^-+\mathbf{4}^-)\mathbf{1}^+]I, \tag{2.2}$$

where, for example, $\mathbf{4}^\pm I(n_1,n_2,n_3,n_4)=I(n_1,n_2,n_3,n_4\pm1)$, etc. Because of the symmetry, we always may assume that $n_4\gtrsim n_3$; the index n_4 can be reduced down to $n_4=1$ by Eq. (2.2). After that, we are left with $I(n_1,n_2,1,1)$, plus trivial integrals which can be expressed in terms of Eq. (2.1).

Next we are going to reduce n_1. To this end, we consider recurrence relations obtained by applying $(\partial/\partial k)\cdot k$ and $(\partial/\partial k)\cdot p$ to the integrand of (1.1):

$$[d-n_1-2n_2-n_4-n_1\mathbf{2}^-\mathbf{1}^+-n_4(\mathbf{2}^--\mathbf{3}^-)\mathbf{4}^+]I=0, \tag{2.3}$$

$$[-2n_1+2n_2+n_4+2n_1(\mathbf{2}^-+2M^2)\mathbf{1}^+-(2n_2+n_4)\mathbf{1}^-\mathbf{2}^+$$
$$+n_4(\mathbf{1}^-\mathbf{3}^-\mathbf{2}^+-\mathbf{1}^-+\mathbf{2}^--\mathbf{3}^-)\mathbf{4}^+]I=0. \tag{2.4}$$

We express $\mathbf{2}^-\mathbf{4}^+I$ from Eq. (2.3) and $\mathbf{1}^-\mathbf{4}^+I$ from its $\mathbf{1}^-\mathbf{2}^+$ shifted version, and substitute both into Eq. (2.4):

$$[d-2n_1-1-(d-n_1-1)\mathbf{1}^-\mathbf{2}^++n_1(\mathbf{2}^-+4M^2)\mathbf{1}^+]I=0. \tag{2.5}$$

Now we express $\mathbf{4}^+I$ from Eq. (2.2) and substitute it into the $\mathbf{2}^+$ shifted version of Eq. (2.3); adding Eq. (2.5) and substituting $n_3=n_4=1$, we arrive at

$$4(M^2-m^2)n_1\mathbf{1}^+I(n_1,n_2,1,1)$$

$$=[-3d+3n_1+2n_2+5+(d-n_1-1)\mathbf{1}^-\mathbf{2}^+$$
$$-4m^2(\mathbf{3}^-\mathbf{4}^++d-n_1-2n_2-3)\mathbf{2}^+]I(n_1,n_2,1,1). \tag{2.6}$$

This relation allows one to reduce n_1 down to $n_1=1$ (note that the term with $\mathbf{3}^-\mathbf{4}^+$ is trivial here). Thus we are left with $I(1,n_2,1,1)$.

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[4]Note that $I(-1,n_2,n_3,n_4)=I(0,n_2-1,n_3,n_4)$.

Finally, substituting $\mathbf{1}^+ I$ from Eq. (2.6) and its shifted version $\mathbf{2}^- \mathbf{1}^+ I$ into Eq. (2.5), we obtain at $n_1 = n_3 = n_4 = 1$

$$[(3d - 2n_2 - 6)\mathbf{2}^- + 4(M^2 + m^2)(2d - 2n_2 - 5)$$
$$+ 16M^2 m^2 (d - 2n_2 - 4)\mathbf{2}^+] I(1, n_2, 1, 1)$$
$$= [(d - 2)\mathbf{1}^- (4m^2 \mathbf{2}^+ + 1) - 4m^2 \mathbf{3}^- \mathbf{4}^+$$
$$\times (4M^2 \mathbf{2}^+ + 1)] I(1, n_2, 1, 1). \tag{2.7}$$

This relation allows one to lower or raise n_2 [note that the integrals on the right-hand side of Eq. (2.7) are trivial]. Therefore, all integrals (1.1) can be expressed via

$$I_0 \equiv I(1, 0, 1, 1), \quad I_1 \equiv I(1, 1, 1, 1), \tag{2.8}$$

and the trivial integrals $T_{0,1}$ (2.1), exactly in d dimensions. The basis integrals (2.8) in d dimensions can be expressed via hypergeometric functions ${}_3F_2$; see Appendix A.

Expressions for $I_{0,1}$ expanded in ε up to the finite terms can be, with some effort, extracted from any two independent integrals out of those calculated in [2–7]. Some details of this procedure are discussed in Appendix B. Introducing a dimensionless variable

$$r \equiv m/M \tag{2.9}$$

and dilogarithmic functions[5]

$$L_+ = -\mathrm{Li}_2(-r) + \tfrac{1}{2} \log^2 r - \log r \log(1 + r) - \tfrac{1}{6} \pi^2$$
$$= \mathrm{Li}_2(-r^{-1}) + \log r^{-1} \log(1 + r^{-1}), \tag{2.10}$$

$$L_- = \mathrm{Li}_2(1 - r) + \tfrac{1}{2} \log^2 r + \tfrac{1}{6} \pi^2 = -\mathrm{Li}_2(1 - r^{-1}) + \tfrac{1}{6} \pi^2$$
$$= -\mathrm{Li}_2(r) + \tfrac{1}{2} \log^2 r - \log r \log(1 - r) + \tfrac{1}{3} \pi^2 \quad (r \leqslant 1)$$
$$= \mathrm{Li}_2(r^{-1}) + \log r^{-1} \log(1 - r^{-1}) \quad (r \geqslant 1), \tag{2.11}$$

the results for the integrals $I_{0,1}$ can be presented as

$$\frac{I_0}{\Gamma^2(1 + \varepsilon)} = -M^{2 - 4\varepsilon} \left[\frac{1}{2\varepsilon^2} + \frac{5}{4\varepsilon} + 2(1 - r^2)^2 (L_+ + L_-) \right.$$
$$\left. - 2 \log^2 r + \frac{11}{8} \right] - m^{2 - 4\varepsilon} \left[\frac{1}{\varepsilon^2} + \frac{3}{\varepsilon} - 2 \log r + 6 \right]$$
$$+ \mathcal{O}(\varepsilon), \tag{2.12}$$

$$\frac{I_1}{\Gamma^2(1 + \varepsilon)} = M^{-4\varepsilon} \left[\frac{1}{2\varepsilon^2} + \frac{5}{2\varepsilon} + 2(1 + r)^2 L_+ + 2(1 - r)^2 L_- \right.$$
$$\left. - 2 \log^2 r + \frac{19}{2} \right] + \mathcal{O}(\varepsilon). \tag{2.13}$$

Equation (2.12) corresponds to a special case of more general results for the sunset-like diagrams presented in [8]. Equation (2.13) is equivalent to the result presented in Eq. (29) of Ref. [6]. In Appendix B, we discuss different ways to derive it. Note that

$$L_+ + L_- = \tfrac{1}{2} \mathrm{Li}_2(1 - r^2) + \log^2 r + \tfrac{1}{12} \pi^2$$
$$= -\tfrac{1}{2} \mathrm{Li}_2(1 - r^{-2}) + \tfrac{1}{12} \pi^2. \tag{2.14}$$

When dealing with diagrams like Fig. 1 with numerators and reducing them to scalar integrals, one cannot directly express $l \cdot p$ via $\mathcal{D}_{1 \ldots 4}$. Therefore, integrals similar to Eq. (1.1) but containing some extra numerators[6] appear. Let us define

$$\tilde{I}(n_1, n_2, n_3, n_4; n_0) \equiv -\frac{1}{\pi^d} \int \frac{d^d k \, d^d l \, \mathcal{N}^{n_0}}{\mathcal{D}_1^{n_1} \mathcal{D}_2^{n_2} \mathcal{D}_3^{n_3} \mathcal{D}_4^{n_4}} \bigg|_{p^2 = M^2}. \tag{2.15}$$

As compared to Eqs. (1.1), it contains an extra factor $\mathcal{N}^{n_0}$ in the numerator, with $n_0 \geqslant 0$ and

$$\mathcal{N} = (2l - k) \cdot p. \tag{2.16}$$

Changing the integration momenta $l \leftrightarrow k - l$, we obtain

$$\tilde{I}(n_1, n_2, n_4, n_3; n_0) = (-1)^{n_0} \tilde{I}(n_1, n_2, n_3, n_4; n_0).$$

In particular, when $n_3 = n_4$ the integrals with odd n_0 vanish. Integrals with $n_{3,4} \leqslant 0$ or $n_1 \leqslant 0$ are trivial, as before.

In principle, one could deal with Eq. (2.15) by generalizing (to higher n_0) the substitutions like $p \cdot l \to (p \cdot k)(k \cdot l)/k^2$ [cf. Eq. (2.2)]. However, the higher n_0 is, the more cumbersome these substitutions are. An alternative, recursive way is to use the integration-by-parts procedure.[7] Applying $(\partial/\partial k) \cdot p$ to the integrand, we obtain the recurrence relation

$$n_4 \mathbf{4}^+ \mathbf{0}^+ \tilde{I} = [-n_1 + n_2 - M^2 n_0 \mathbf{0}^- + n_1 (\mathbf{2}^- + 2M^2) \mathbf{1}^+$$
$$- n_2 \mathbf{1}^- \mathbf{2}^+ - \tfrac{1}{2} n_4 (\mathbf{1}^- - \mathbf{2}^-) \mathbf{4}^+] \tilde{I}. \tag{2.17}$$

We may assume $n_4 \geqslant n_3$; if $n_4 > 1$, then both n_0 and n_4 can be reduced by Eq. (2.17). Thus we are left with already known integrals and $\tilde{I}(n_1, n_2, 1, 1; n_0)$ with even n_0. Applying $(\partial/\partial k) \cdot l$ to the integrand, we obtain at $n_3 = n_4 = 1$

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[5]We adopt the notation $L_\pm$ which has been used in [3–5] (cf., e.g., Eq. (A3) of [5]). In [6], the functions L_- and L_+ were called R_1 and R_2, respectively. In [8] the functions $T^+(r) = -2L_- + \tfrac{1}{2} \log^2 r + \tfrac{1}{3} \pi^2$ and $T^-(r) = -2L_+ + \tfrac{1}{2} \log^2 r - \tfrac{1}{6} \pi^2$ were used. They have a nice property $T^\pm(r^{-1}) = -T^\pm(r)$, and they are analytic continuations of each other at $r \leftrightarrow -r$.

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[6]These numerators are not really irreducible, like e.g. in the three-point functions [27], since the corresponding integrals can be reduced algebraically.

[7]A similar algorithm has been constructed in [10].

$$n_1 \mathbf{1}^+ \mathbf{0}^+ \tilde{I}(n_1, n_2, 1, 1; n_0)$$

$$= [-\tfrac{1}{4} n_0 (\mathbf{1}^- - \mathbf{2}^-) \mathbf{0}^- + 2 n_1 \mathbf{3}^- \mathbf{1}^+ + 2 n_2 \mathbf{3}^- \mathbf{2}^+$$

$$+ (\mathbf{2}^- - \mathbf{3}^- + 2m^2) \mathbf{4}^+] \tilde{I}(n_1, n_2, 1, 1; n_0). \qquad (2.18)$$

If $n_1 > 1$, then both n_0 and n_1 can be reduced by Eq. (2.18). Now, we are left with known integrals and $\tilde{I}(1, n_2, 1, 1; n_0)$ with even n_0. Finally, applying $(\partial/\partial l) \cdot l$ to the integrand, we obtain at $n_1 = n_3 = n_4 = 1$

$$(d + n_0 - 3) \tilde{I}(1, n_2, 1, 1; n_0)$$

$$= -(\mathbf{2}^- - \mathbf{3}^- + 4m^2) \mathbf{4}^+ \tilde{I}(1, n_2, 1, 1; n_0). \qquad (2.19)$$

The right-hand side contains integrals with the same n_0 but $n_4 > 1$; in them, n_0 can be again reduced by Eq. (2.17). In this way, these more general integrals (2.15) are also reduced to the basis ones, $I_{0,1}$ and $T_{0,1}$.

III. EFFECT OF m_c ON b QUARK CHROMOMAGNETIC INTERACTION

First, we calculate contributions of a flavor with mass m (e.g., the c quark) to the mass and wave function renormalization constants of a quark with mass M (e.g., the b quark) in the on-shell scheme (Fig. 1) in d dimensions. The results are[8]

$$\Delta Z_M = T_F C_F \frac{g_0^4}{(4\pi)^d} \left[-\frac{(d-2)^2(d-5)}{(d-3)(d-6)M^4} \left(\frac{T_0}{2r^2} + T_1 \right) \right.$$

$$\left. - \frac{d^2 - 9d + 16}{(d-6)M^2} I_0 + \frac{4(d-5)(1+r^2)}{d-6} I_1 \right], \qquad (3.1)$$

$$\Delta Z_Q = T_F C_F \frac{g_0^4}{(4\pi)^d} \frac{d-1}{1-r^2} \sum_{i=1}^{4} a_i J_i, \qquad (3.2)$$

where

$$J_1 = \frac{(d-2)^2 T_0}{(d-3)(d-5)(d-6)M^2 m^2},$$

$$J_2 = \frac{(d-2)^2 T_1}{(d-1)(d-3)(d-6)M^4},$$

$$J_3 = \frac{I_0}{(d-1)(d-6)M^2}, \quad J_4 = \frac{I_1}{(d-1)(d-6)}.$$

The results for the coefficients a_i in ΔZ_Q are

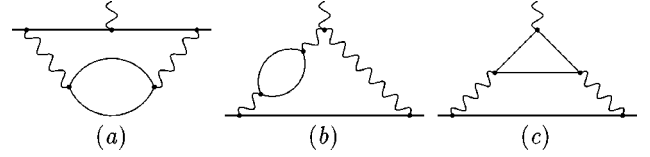[8] In QCD, $T_F = \frac{1}{2}$, $C_F = (N^2 - 1)/(2N)$, $C_A = N$, where $N = 3$ is the number of colors.

FIG. 3. Diagrams for the chromomagnetic moment: (b) implies also the mirror symmetric diagram, and (c) has two orientations of the quark loop.

$$a_1 = (2d^4 r^4 - d^4 r^2 - 39 d^3 r^4 + 21 d^3 r^2 - 2d^3 + 272 d^2 r^4$$

$$- 159 d^2 r^2 + 30 d^2 - 789 dr^4 + 509 dr^2 - 144 d + 770 r^4$$

$$- 566 r^2 + 216)/[4(d-2)(d-7)r^2],$$

$$a_2 = (d^2 + 5dr^2 - 12d - 25 r^2 + 31)/2,$$

$$a_3 = (2d^3 r^2 + d^3 - 17 d^2 r^2 - 12 d^2 + 27 dr^2 + 47 d$$

$$+ 8 r^2 - 56)/2,$$

$$a_4 = 2(d^3 r^2 - 12 d^2 r^2 + d^2 - 5 dr^4 + 45 dr^2 - 6d$$

$$+ 25 r^4 - 54 r^2 + 5).$$

After expansion in ε up to finite terms, these formulas reproduce the results of [3,4].

In the case $m = 0$,

$$T_0 = T_1 = 0, \quad I_1 = -\frac{3d-8}{4(2d-7)} M^2 I_0. \qquad (3.3)$$

Note that I_0 at $m = 0$ is called $-M^{2d-6} I_1$ in [24]. ΔZ_Q contains a term $J_1/r^2 \sim T_0/m^4$ which does not vanish in the limit $m \to 0$. This means that there is no smooth $m \to 0$ limit in ΔZ_Q [4]. Omitting the $T_{0,1}$ terms and then setting $m = 0$, we reproduce the results of [3,4] for the massless quark contributions, exactly in d dimensions.

In the case $m \to M$,

$$T_1 = T_0 \left[1 + \frac{d-2}{2} (1 - r^2) \right] + \mathcal{O}((1 - r^2)^2),$$

$$I_1 = -\frac{3d-8}{4(d-4)M^2} I_0 \left[1 + \frac{1-r^2}{2} \right] - \frac{3(d-2)^2}{8(d-3)(d-4)M^4}$$

$$\times T_0 \left[1 + \frac{2d+3}{6} (1 - r^2) \right] + \mathcal{O}((1 - r^2)^2). \qquad (3.4)$$

Note that T_0 and I_0 at $m = M$ are called in [24] $M^{2d-4} I_0^2$ and $-M^{2d-6} I_2$, respectively. Using this, we reproduce the results of [3,4] for the contributions of the quark with mass M, exactly in d dimensions.

Now we calculate the contribution $\Delta \mu_g$ of a flavor with mass m (say, c) to the chromomagnetic moment of a quark with mass M (say, b) (Fig. 3), taking into account the wave function renormalization (3.1). We use the background field method [28]. Discontinuity in the limit $m \to 0$ in the C_F term has cancelled with Z_Q, but in the C_A term it is still here. We

reproduce the $m=0$ result (by omitting $T_{0,1}$ terms) and the $m=M$ result from the Appendix in [24], exactly in d dimensions.[9]

Combining this result with the corresponding HQET term $\Delta\tilde{\mu}_{g0}$ [24], we obtain

$$\Delta\mu_g - \Delta\tilde{\mu}_{g0} = T_F \frac{g_0^4}{(4\pi)^d} \frac{1}{1-r^2}\left[C_F\sum_{i=1}^{4} a_{Fi}J_i + C_A\sum_{i=1}^{4} a_{Ai}J_i \right],$$
(3.5)

with the coefficients

$$a_{F1} = (d^3r^2 - 11d^2r^2 - 3d^2 + 33dr^2 + 30d - 19r^2 - 71)/2,$$

$$a_{F2} = -(d^3r^2 - 2d^3 - 17d^2r^2 + 27d^2 + 89dr^2 - 116d - 133r^2 + 151),$$

$$a_{F3} = d^4r^2 + 2d^4 - 7d^3r^2 - 31d^3 - 7d^2r^2 + 168d^2 + 105dr^2 - 375d - 152r^2 + 296,$$

$$a_{F4} = 4(d^4r^2 + d^3r^4 - 15d^3r^2 - 17d^2r^4 + 85d^2r^2 - d^2 + 89dr^4 - 221dr^2 + 6d - 133r^4 + 210r^2 - 5),$$

$$a_{A1} = -(d^3 - d^2r^2 - 13d^2 + 11dr^2 + 52d - 26r^2 - 64)/8,$$

$$a_{A2} = (2d^3r^2 - 3d^3 - 29d^2r^2 + 39d^2 + 129dr^2 - 156d - 182r^2 + 200)/4,$$

$$a_{A3} = -(3d-8)(d^3 + d^2r^2 - 11d^2 - 11dr^2 + 38d + 26r^2 - 44)/4,$$

$$a_{A4} = -(d^4r^2 + 2d^3r^4 - 18d^3r^2 + 2d^3 - 29d^2r^4 + 114d^2r^2 - 18d^2 + 129dr^4 - 299dr^2 + 44d - 182r^4 + 282r^2 - 28).$$

Discontinuity in the limit $m\to0$ in the C_A term has cancelled. The difference of this contribution and that of a massless flavor is finite as $\varepsilon\to0$; it represents the amount by which the b quark chromomagnetic interaction coefficient changes due to a nonzero mass of c quark:[10]

$$\Delta C_m = T_F\left(\frac{\alpha_s}{4\pi}\right)^2\left[8C_F A_F + \frac{4}{3}C_A A_A \right],$$
(3.6)

$$A_F = -r(1+r)(1-r-4r^2)L_+ + r(1-r)(1+r-4r^2)L_- + 6r^2\left(\log r + \frac{4}{3}\right),$$

_____

[9]There is a typo in the Appendix of [24]: $J_{0,1,2}$ should read $J_{1,2,3}$.
[10]In the notations of [5], $A_F = \Delta_2 - 4\Delta_3$.

$$A_A = -(1+r)(2+4r-r^2)L_+ - (1-r)(2-4r-r^2)L_- + 2\log^2 r + \frac{1}{3}\pi^2 + 2r^2(\log r + 1).$$

It vanishes as $r\to0$, whereas at $r=1$ it reproduces the result of Ref. [24] (namely, the one with heavy quark loop).

In the QED case ($T_F=1$, $C_F=1$, $C_A=0$, $\alpha_s\to\alpha$), the formula (3.6) is closely related to the contribution of a lepton with mass m (e.g., electron or τ) to the magnetic moment of a lepton with mass M (e.g., muon). Adding back the massless contribution [which was subtracted in Eq. (3.6)] and reexpressing the modified minimal subtraction scheme ($\overline{\text{MS}}$) coupling in the one-loop contribution via the on-shell coupling $\alpha(M) = \alpha[1 - (2\alpha/3\pi)\log r]$, we find that this contribution is given by Eq. (3.6) with $A_F\to A_F - \frac{2}{3}\log r - \frac{25}{18}$. It was calculated in [2]; our result agrees with it (and has a simpler form).

Expanding Eq. (3.6) at $r\ll1$ and $r\gg1$, we obtain

$$A_F = \frac{1}{2}\pi^2 r + 2(4\log r + 3)r^2 - \frac{5}{2}\pi^2 r^3 + 2(2\log^2 r - \frac{14}{3}\log r$$
$$+ \frac{1}{3}\pi^2 + \frac{44}{9})r^4 + \sum_{n=3}^{\infty}\left(2g_F(2n)\log r + \frac{dg_F(2n)}{dn} \right)r^{2n}$$
$$= \sum_{n=0}^{\infty}\left(-2g_F(-2n)\log r + \frac{dg_F(-2n)}{dn} \right)r^{-2n},$$

$$A_A = 3\pi^2 r + 3(\log^2 r + 4\log r + \frac{1}{6}\pi^2 - 3)r^2 - \frac{1}{2}\pi^2 r^3$$
$$+ \sum_{n=2}^{\infty}\left(2g_A(2n)\log r + \frac{dg_A(2n)}{dn} \right)r^{2n}$$
$$= 2\log^2 r + \frac{43}{3}\log r + \frac{1}{3}\pi^2 + \frac{239}{18}$$
$$+ \sum_{n=1}^{\infty}\left(-2g_A(-2n)\log r \right.$$
$$\left. + \frac{dg_A(-2n)}{dn} \right)r^{-2n},$$

$$g_F(x) = \frac{1}{x-1} - \frac{5}{x-3} + \frac{4}{x-4},$$

$$g_A(x) = -\frac{2}{x} + \frac{6}{x-1} + \frac{3}{x-2} - \frac{1}{x-3}.$$

Note that loop effects of a very heavy flavor (say, t) do not decouple in C_m, because it is not directly measurable. Adding $(-\frac{2}{3}\log r - \frac{25}{18})$ to the expansions of A_F, we obtain the expansions of the contribution of a lepton with mass m to the magnetic moment of a lepton with mass M in QED; they agree with the expansions of [2] obtained in [29]. It is also easy to obtain as many terms as needed in the expansions as

$r \to 1$ using the formulas for L_- via $1-r$ or $1-r^{-1}$ [see Eq. (2.11)] and for $L_+ + L_-$ via $1-r^2$ or $1-r^{-2}$ [see Eq. (2.14)].

IV. CONCLUSIONS

We have presented an algorithm which reduces any scalar integral $\tilde{I}(n_1, n_2, n_3, n_4; n_0)$ [given by Eq. (2.15)] to two trivial basis integrals (2.1) and two nontrivial ones, whose ε expansions up to the finite terms are given by Eqs. (2.12) and (2.13). The algorithm has been implemented as a REDUCE package. It has been tested by evaluating about 27000 recurrence relations for specific values of n_i, including those relations which were not directly used for construction of the algorithm. This package allows one to calculate, completely automatically, the on-shell two-loop self-energy diagrams (Fig. 1) in any theory with two massive particles having different masses.[11]

We calculated the effect of nonzero m_c on the b quark chromomagnetic interaction coefficient (3.6). Combining it with the result for $m_c = 0$ [24], we have numerically

$$C_{\mathrm{m}}(m_b) \simeq 1 + \frac{13}{6} \frac{\alpha_s(m_b)}{\pi} + (14.1439 + \Delta_{\mathrm{m}}(m_c/m_b)) \left(\frac{\alpha_s}{\pi} \right)^2,$$

$$\Delta_{\mathrm{m}}(0.3 \pm 0.03) = 0.98 \pm 0.07. \tag{4.1}$$

The effect is moderate and positive. This is the only nontrivial coefficient in the HQET Lagrangian up to $1/m_b$ level.

Due to reparametrization invariance [30], it also determines [31] the spin-orbit interaction coefficient—the most important spin-symmetry breaking coefficient at $1/m_b^2$:

$$C_{\mathrm{so}}(\mu) = 2 C_{\mathrm{m}}(\mu) - 1. \tag{4.2}$$

This relation has a very simple physical interpretation.[12] The spin-orbit interaction is the interaction of a moving (nonrelativistic) heavy quark with chromoelectric field. In the quark rest frame, the field acquires a chromomagnetic component. The quark chromomagnetic moment interacts with it, producing the term $2C_{\mathrm{m}}$. There is also Thomas precession, which compensates half of the chromomagnetic contribution at the tree level. This is a purely kinematic effect due to Lorentz transformations, and hence it gets no corrections.

The influence of m_c on other $1/m_b^2$ terms in the b quark HQET Lagrangian, as well as on $1/m_b$ expansions of various QCD currents, can also be calculated using the presented method.

APPENDIX A: INTEGRALS IN d DIMENSIONS

Here we present the most general results for the integrals (1.1). To indicate that the results of this appendix are valid for arbitrary powers of propagators, not only the integer ones, we substitute $n_i \to \nu_i$.

Using Mellin-Barnes contour integral representation for the massive denominator $\mathcal{D}_1^{-\nu_1}$ (for details, see in [32]), we get

$$I(\nu_1, \nu_2, \nu_3, \nu_4) = \frac{M^{d-2\nu_1-2\nu_2} m^{d-2\nu_3-2\nu_4}}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(\nu_4)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \left(\frac{M^2}{m^2} \right)^s \Gamma(-s) \Gamma\left(\nu_1 + \nu_2 - \frac{d}{2} - s \right)$$

$$\times \frac{\Gamma(\nu_3+s) \Gamma(\nu_4+s) \Gamma\left(\nu_3 + \nu_4 - \frac{d}{2} + s \right) \Gamma(d - \nu_1 - 2\nu_2 + 2s)}{\Gamma(\nu_3 + \nu_4 + 2s) \Gamma(d - \nu_1 - \nu_2 + s)},$$

where the integration contour separates right and left series of poles of Γ functions occurring in the numerator of the integrand.

If we close the contour to the right, we should sum over two series of poles.[13] The general result can be presented as

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[11]The package may be obtained from http://www.inp.nsk.su/~grozin/

[12]A.G.G. thanks I.B. Khriplovich for this interpretation.

[13]If we close the contour to the left, we get a more cumbersome result in terms of the hypergeometric series of m^2/M^2, corresponding to the analytic continuation of Eq. (A1).

$$I(\nu_1,\nu_2,\nu_3,\nu_4)=(m^2)^{d-\nu_1-\nu_2-\nu_3-\nu_4}\left\{\frac{\Gamma\left(\frac{d}{2}-\nu_1-\nu_2\right)\Gamma(\nu_1+\nu_2+\nu_3+\nu_4-d)\Gamma\left(\nu_1+\nu_2+\nu_3-\frac{d}{2}\right)\Gamma\left(\nu_1+\nu_2+\nu_4-\frac{d}{2}\right)}{\Gamma(\nu_3)\Gamma(\nu_4)\Gamma\left(\frac{d}{2}\right)\Gamma(2\nu_1+2\nu_2+\nu_3+\nu_4-d)}\right.$$

$$\times {}_5F_4\left(\begin{array}{c}\nu_1+\nu_2+\nu_3-\frac{d}{2},\nu_1+\nu_2+\nu_4-\frac{d}{2},\nu_1+\nu_2+\nu_3+\nu_4-d,\frac{1}{2}\nu_1,\frac{1}{2}(\nu_1+1)\\[4pt]\nu_1+\nu_2+\frac{1}{2}(\nu_3+\nu_4)-\frac{d}{2},\nu_1+\nu_2+\frac{1}{2}(\nu_3+\nu_4+1)-\frac{d}{2},\nu_1+\nu_2-\frac{d}{2}+1,\frac{d}{2}\end{array}\middle|\frac{M^2}{m^2}\right)$$

$$+\left(\frac{M^2}{m^2}\right)^{d/2-\nu_1-\nu_2}\frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)\Gamma\left(\nu_3+\nu_4-\frac{d}{2}\right)\Gamma(d-\nu_1-2\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_3+\nu_4)\Gamma(d-\nu_1-\nu_2)}$$

$$\left.\times {}_5F_4\left(\begin{array}{c}\nu_3,\nu_4,\nu_3+\nu_4-\frac{d}{2},\frac{1}{2}(d-\nu_1-2\nu_2),\frac{1}{2}(d-\nu_1-2\nu_2+1)\\[4pt]\frac{1}{2}(\nu_3+\nu_4),\frac{1}{2}(\nu_3+\nu_4+1),d-\nu_1-\nu_2,\frac{d}{2}-\nu_1-\nu_2+1\end{array}\middle|\frac{M^2}{m^2}\right)\right\},\tag{A1}$$

where

$$_PF_Q\left(\begin{array}{c}a_1,\ldots,a_P\\c_1,\ldots,c_Q\end{array}\middle|z\right)=\sum_{j=0}^{\infty}\frac{(a_1)_j\ldots(a_P)_j}{(c_1)_j\ldots(c_Q)_j}\frac{z^j}{j!}$$

is the generalized hypergeometric function; $(a)_j\equiv\Gamma(a+j)/\Gamma(a)$ is the Pochhammer symbol.

For trivial cases $\nu_1=0$ and $\nu_3=0$ (or $\nu_4=0$), the expression (A1) reproduces known results in terms of Γ functions. When $\nu_2=0$ (the ''sunset'' configuration), the same result (A1) can be reproduced in this limit (i.e., $p^2=M^2$ and two other masses are equal) by taking two sums in the hypergeometric series of three variables presented in Ref. [33].

One can see that for integer (non-negative) values of ν_i both $_5F_4$ functions in Eq. (A1) reduce, for arbitrary d, to a finite sum of $_3F_2$ functions of the same argument. For example, we get for our basis integrals $I_0=I(1,0,1,1)$ and $I_1=I(1,1,1,1)$:

$$\frac{I_0}{\Gamma^2(1+\varepsilon)}=-\frac{(m^2)^{1-2\varepsilon}}{\varepsilon^2(1-\varepsilon)}\left\{\frac{1}{1-2\varepsilon}\,_3F_2\left(\begin{array}{c}1,\frac{1}{2},-1+2\varepsilon\\2-\varepsilon,\frac{1}{2}+\varepsilon\end{array}\middle|\frac{M^2}{m^2}\right)+\left(\frac{M^2}{m^2}\right)^{1-\varepsilon}\,_3F_2\left(\begin{array}{c}1,\varepsilon,\frac{3}{2}-\varepsilon\\3-2\varepsilon,\frac{3}{2}\end{array}\middle|\frac{M^2}{m^2}\right)\right\},$$

$$\frac{I_1}{\Gamma^2(1+\varepsilon)}=-\frac{(m^2)^{-2\varepsilon}}{\varepsilon^2}\left\{\frac{1}{2(1-\varepsilon)(1+2\varepsilon)}\,_3F_2\left(\begin{array}{c}1,\frac{1}{2},2\varepsilon\\2-\varepsilon,\frac{3}{2}+\varepsilon\end{array}\middle|\frac{M^2}{m^2}\right)-\left(\frac{M^2}{m^2}\right)^{-\varepsilon}\frac{1}{1-2\varepsilon}\,_3F_2\left(\begin{array}{c}1,\varepsilon,\frac{1}{2}-\varepsilon\\2-2\varepsilon,\frac{3}{2}\end{array}\middle|\frac{M^2}{m^2}\right)\right\},$$

exactly in $d=4-2\varepsilon$ dimensions.[14] Expansion of I_0 and I_1 in ε up to finite terms is given in Eqs. (2.12),(2.13).

APPENDIX B: VARIOUS WAYS TO CALCULATE BASIS INTEGRALS

Using a procedure similar to [7], the result (2.13) can be derived as follows. Note that $m^{4\varepsilon}I_1$ and $m^{4\varepsilon}I(2,0,1,1)$ are dimensionless and depend only on $r=m/M$. Differentiating the definition (1.1) in M and taking into account $p=Mv$, we see that I_1 should obey the differential equation[15]

$$r^2(d/dr)[r^{-1}m^{4\varepsilon}I_1]=-m^{4\varepsilon}I(2,0,1,1).\tag{B1}$$

Therefore,

$$m^{4\varepsilon}I_1=r\int_r^{\infty}\frac{d\rho}{\rho^2}m^{4\varepsilon}I(2,0,1,1)|_{r\to\rho}.\tag{B2}$$

The expression for $I(2,0,1,1)$ can be taken, e.g., from Ref. [8]:

$$\frac{I(2,0,1,1)}{\Gamma^2(1+\varepsilon)}=(M^2)^{-2\varepsilon}\left[\frac{1}{2\varepsilon^2}+\frac{1}{2\varepsilon}-\frac{1}{2}-2\log^2 r+2(1-r^2)\right.$$

$$\left.\times(L_++L_-)\right]+\mathcal{O}(\varepsilon).\tag{B3}$$

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[14]If $M=m$, one arrives at a representation in terms of $_3F_2$ functions of the unit argument [11].

[15]This equation can also be derived using the method of [34].

Substituting Eq. (B3) into Eq. (B2) we get Eq. (2.13).

An alternative derivation follows the method used in [6]. The integrals (1.1) can be presented as

$$I(n_1,n_2,n_3,n_4) = -\frac{i}{\pi^{d/2}} \int \frac{\Pi_{12}(k^2)\, d^d k}{\mathcal{D}_1^{n_1}\mathcal{D}_2^{n_2}} \Bigg|_{p^2=M^2},$$

$$\Pi_{n_3 n_4}(k^2) \equiv -\frac{i}{\pi^{d/2}} \int \frac{d^d l}{\mathcal{D}_3^{n_3}\mathcal{D}_4^{n_4}}. \quad \text{(B4)}$$

For instance, when we consider integrals with $n_3=1$ and $n_4=2$ we get

$$\Pi_{12}(k^2) = \frac{\lambda^2-1}{4m^2\lambda} \log\frac{\lambda+1}{\lambda-1} + \mathcal{O}(\varepsilon), \quad \lambda \equiv \sqrt{1-\frac{4m^2}{k^2}}. \quad \text{(B5)}$$

Moreover, we know that $\Pi_{12}(k^2)$ may be represented via dispersion integral, $\Pi_{12}(k^2) = \int ds\,\rho(s)/(s-k^2)$, the explicit form of the spectral density $\rho(s)$ being not relevant for our discussion.

As the first example, let us consider the integral $I(2,0,1,2)$, which is convergent. Substituting the spectral representation for Π_{12}, combining the denominators $\mathcal{D}_1^2$ and $(s-k^2)$ by the Feynman formula and, then, calculating the integrals over k and s, we arrive at

$$I(2,0,1,2) = \int_0^1 dx \frac{x}{1-x} \Pi_{12}(k^2) \Bigg|_{k^2=-M^2 x^2/(1-x)} + \mathcal{O}(\varepsilon). \quad \text{(B6)}$$

If we consider, say, $I(3,0,1,2)$, we get in the integrand $(d\Pi_{12}(k^2)/dk^2)|_{k^2=-M^2 x^2/(1-x)}$. The second example is $I(2,1,1,2)$. In this case, the on-shell singularity can be separated via $\Pi_{12}(k^2) = [\Pi_{12}(k^2)-\Pi_{12}(0)] + \Pi_{12}(0)$. Note that $[\Pi_{12}(k^2)-\Pi_{12}(0)]/k^2 = \int ds\,\rho(s)/[s(s-k^2)]$. In this way, we obtain

$$\frac{I(2,1,1,2)}{\Gamma^2(1+\varepsilon)} = -\frac{(Mm)^{-2-2\varepsilon}}{4\varepsilon} - \int_0^1 dx \frac{x}{1-x}$$

$$\times \frac{\Pi_{12}(k^2)-\Pi_{12}(0)}{k^2} \Bigg|_{k^2=-M^2 x^2/(1-x)} + \mathcal{O}(\varepsilon). \quad \text{(B7)}$$

Integrals with $n_2>1$ require more subtractions, and those with $n_1>2$ involve derivatives.

After inserting the expression (B5), the integrals over x are of the following form:

$$\int_0^1 \left[\frac{A(x)}{\lambda(x)} \log\frac{\lambda(x)+1}{\lambda(x)-1} + B(x) \right] dx, \quad \text{(B8)}$$

where $\lambda(x) \equiv \lambda|_{k^2=-M^2 x^2/(1-x)}$ [cf. Eq. (B5)], while $A(x)$ and $B(x)$ are rational functions. Introducing the new variable y (the limits of the y integration are also from 0 to 1),

$$y = \frac{x(\lambda(x)-1)}{2r}, \quad x = \frac{r(1-y^2)}{y+r},$$

$$\frac{1}{\lambda(x)} \frac{dx}{dy} = -\frac{r(1-y^2)}{(y+r)^2}, \quad \text{(B9)}$$

we see that all occurring structures become rational, since

$$\lambda(x) = \frac{1+2ry+y^2}{1-y^2}, \quad \frac{\lambda(x)+1}{\lambda(x)-1} = \frac{1+ry}{y(r+y)}. \quad \text{(B10)}$$

Therefore, the integral (B8) can be expressed in terms of dilogarithms. In this way, from Eq. (B6), (B7) we obtain

$$I(2,0,1,2) = \frac{L_+ + L_-}{M^2} + \mathcal{O}(\varepsilon),$$

$$\frac{I(2,1,1,2)}{\Gamma^2(1+\varepsilon)} = \frac{1}{4M^2 m^{2+4\varepsilon}} \left[-\frac{1}{\varepsilon} + r(L_+-L_-) + 2 \right] + \mathcal{O}(\varepsilon). \quad \text{(B11)}$$

This is enough to reproduce Eqs. (2.12), (2.13).

Similar procedure can also be applied to the integrals $I(n_1,n_2,n_3,n_4)$ with other values of n_i, as soon as a suitable subtraction can be performed. Apart from subtracting $\Pi_{n_3 n_4}(0)$ (and higher terms of the Taylor expansion in k^2), subtractions of $\Pi_{n_3 n_4}|_{m=0}$ may also be considered.

Some further results of interest are

$$I(1,0,2,2) = -2\frac{L_+ + L_-}{M^2} - \frac{L_+ - L_-}{m^2} + \mathcal{O}(\varepsilon),$$

$$I(2,0,2,2) = \frac{1}{4M^2 m^2} \left[r(L_+-L_-) + 2 - \frac{2r^2 \log r}{1-r^2} \right] + \mathcal{O}(\varepsilon).$$

In particular, if we take $I(2,0,1,2)$ and $I(1,0,2,2)$ as independent finite integrals, we get only "pure" $L_+ \pm L_-$ combinations, without logarithms, etc.

The results (2.12) and (2.13) can also be checked by using the formulas of Appendix A in [5]. The quantities $\Delta_{1,2,3}$ were calculated there, which are certain combinations of the integrals

$$(M^2)^{2-n_1-n_2} J(n_1,n_2) \equiv -\frac{i}{\pi^{d/2}} \int \frac{P(k^2,m)-P(k^2,0)}{\mathcal{D}_1^{n_1}\mathcal{D}_2^{n_2}} d^d k$$

$$= I(n_1,n_2,1,1)|_{m=0} - I(n_1,n_2,1,1)$$

$$+ 2I(n_1,n_2+1,1,0)$$

$$+ \frac{4m^2}{d-2} I(n_1,n_2+1,1,1), \quad \text{(B12)}$$

where

$$P(k^2,m)=-\left(1+\frac{4m^2}{(d-2)k^2}\right)\Pi_{11}(k^2)-\frac{2}{k^2}\Pi_{10}$$

$$=m^{-2\varepsilon}\Gamma(1+\varepsilon)\left[-\frac{1}{\varepsilon}+\left(1+\frac{2m^2}{k^2}\right)\right.$$

$$\left.\times\left(\lambda\log\frac{\lambda+1}{\lambda-1}-2\right)\right]+\mathcal{O}(\varepsilon)\qquad\text{(B13)}$$

is proportional to the fermion-loop contribution to the gluon polarization operator.[16] When an integral contains neither ultraviolet nor infrared (on-shell) singularities, it can be calculated in four dimensions. Averaging $1/\mathcal{D}_1^{n_1}$ over the directions of k in four-dimensional Euclidean space, we obtain $A_{n_1}/\mathcal{D}_2^{n_1}$, where $A_0=A_{-1}=1$, $A_1=x$, $A_2=x^2/(2-x)$, and $k^2=-M^2x^2/(1-x)$. Therefore, convergent integrals are given by

$$J(n_1,n_2)=\int_0^1\Phi_{n_1n_2}(x)[P(k^2,m)$$

$$-P(k^2,0)]_{k^2=-M^2x^2/(1-x)}dx,\qquad\text{(B14)}$$

$$\Phi_{n_1n_2}(x)=A_{n_1}(x)x^{3-2n_1-2n_2}(1-x)^{n_1+n_2-3}(2-x).$$

In terms of these integrals, the quantities $\Delta_{1,2,3}$ used in [5] can be expressed as

$$\Delta_1=\tfrac{1}{6}[2J(1,1)+J(2,0)],\quad\Delta_2=\tfrac{1}{3}[J(1,1)-J(2,0)],$$

$$\Delta_3=\tfrac{1}{6}[J(0,1)-J(1,0)-J(2,0)].\qquad\text{(B15)}$$

This coincides with the integral representations [5]. Explicit expressions for Δ_i given in Eq. (A2) of [5] [which can be obtained using the substitution (B9)] provide three independent checks on our basis integrals. Note that the integral calculated in [3] is $\Delta=\tfrac{1}{4}(\Delta_1+\Delta_3)$.

The integrals[17]

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[16]In Refs. [3,5] the notation $\Pi(-m^2/k^2)\equiv[P(k^2,m)-P(k^2,0)]_{d=4}$ has been used. To get the d-dimensional unrenormalized contribution to the gluon polarization operator, Eq. (B13) should be multiplied by $-2T_Fg_0^2(4\pi)^{-d/2}(d-2)/(d-1)$.

[17]An integral of this type was considered in [4], where the notation $\bar{\Pi}(k^2)\equiv[P(k^2,m)-P(0,m)]_{d=4}$ was used.

$$(M^2)^{2-n_1-n_2}\bar{J}(n_1,n_2)\equiv-\frac{i}{\pi^{d/2}}\int\frac{P(k^2,m)-P(0,m)}{\mathcal{D}_1^{n_1}\mathcal{D}_2^{n_2}}d^dk$$

$$=\frac{2(d-1)}{3(d-2)}I(n_1,n_2,2,0)$$

$$-I(n_1,n_2,1,1)+2I(n_1,n_2+1,1,0)$$

$$+\frac{4m^2}{d-2}I(n_1,n_2+1,1,1)\qquad\text{(B16)}$$

have better infrared convergence but worse ultraviolet convergence. We define $\bar{\Delta}_{1,2,3}$ via

$$\bar{J}(2,1)=\bar{\Delta}_1,\quad\bar{J}(2,0)-\bar{J}(1,1)=\bar{\Delta}_2,\qquad\text{(B17)}$$

$$\bar{J}(0,1)-\bar{J}(1,0)-\bar{J}(1,1)=\bar{\Delta}_3+\frac{\varepsilon}{2-\varepsilon}\bar{J}(0,2).\quad\text{(B18)}$$

The combination of integrals on the left-hand side of Eq. (B18) is rather interesting. As $k\to\infty$, the combination of their denominators behaves as

$$\frac{1}{M^2\mathcal{D}_2}-\frac{1}{M^2\mathcal{D}_1}-\frac{1}{\mathcal{D}_1\mathcal{D}_2}\Rightarrow\frac{\varepsilon}{(2-\varepsilon)k^4}+\mathcal{O}\left(\frac{1}{k^6}\right),$$

where (on the right-hand side) terms vanishing when averaged over the directions of k are omitted. The term $\mathcal{O}(1/k^6)$ gives an ultraviolet-convergent integral. The term $\mathcal{O}(\varepsilon/k^4)$ might have been omitted, if an arbitrarily large ultraviolet cutoff were considered. Without the cutoff, it gives a purely ultraviolet contribution (similar to the axial anomaly) $\varepsilon\bar{J}(0,2)/(2-\varepsilon)$. The term $\bar{\Delta}_3$ can be calculated at $d=4$ using the convergent integral representation with $\Phi_{01}(x)-\Phi_{10}(x)-\Phi_{11}(x)$. Using Eq. (B9), we obtained

$$\bar{\Delta}_1=\tfrac{1}{4}(1+r)(4-r+r^2)L_++\tfrac{1}{4}(1-r)(4+r+r^2)L_-$$

$$+(\tfrac{5}{3}+\tfrac{1}{2}r^2)\log r+\tfrac{14}{9}+\tfrac{1}{2}r^2,$$

$$\bar{\Delta}_2=3r(1-r^2)(L_+-L_-)+2(2-3r^2)\log r+\tfrac{16}{3}-6r^2,$$

$$\bar{\Delta}_3=3r(1+r)^2(1-2r)L_+-3r(1-r)^2(1+2r)L_-$$

$$-3(1-4r^2)\log r+\tfrac{19}{4}-15r^2$$

and checked that these expressions agree with d-dimensional results obtained by our program as $d\to4$. The integral calculated in [4] is $\bar{\Delta}=\tfrac{1}{16}(4\bar{\Delta}_1+3\bar{\Delta}_2-2\bar{\Delta}_3)$. Equations (2.12) and (2.13) can be reconstructed from any two out of four independent integrals from [3–5].

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